The Full Potential Equation :

- Used to look at effect of linear compressibility between ~ $M = 0.3 - 0.7$

→ inviscid, compressible flow in a uniform stream which is isentropic & irrotational.

Velocity of irrotational flow written as :

$$u = \nabla \phi \longrightarrow u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$$

→ We want to obtain equation for ϕ, using continuity, momentum & isentropic speed of sound, derivation in slides (non-examinable), to give :

$$\left[a^2 - \left(\frac{\partial \phi}{\partial x}\right)^2\right]\frac{\partial^2 \phi}{\partial x^2} + \left[a^2 - \left(\frac{\partial \phi}{\partial y}\right)^2\right]\frac{\partial^2 \phi}{\partial y^2} - 2\left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right)\frac{\partial^2 \phi}{\partial x \partial y} = 0 \qquad (1)$$

unknowns are a & ϕ

also have

$$a^2 = a_0^2 - \frac{\gamma - 1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] \qquad (2)$$

where a_0 is known property of flow

Comparing equation (1) to Laplace's Equation :

$$\left[a^2 - \left(\frac{\partial \phi}{\partial x}\right)^2\right]\frac{\partial^2 \phi}{\partial x^2} + \left[a^2 - \left(\frac{\partial \phi}{\partial y}\right)^2\right]\frac{\partial^2 \phi}{\partial y^2} - 2\left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right)\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

Non Linear

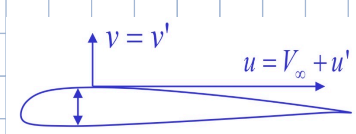
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Linear

- can't solve full potential with superposition because it's non-linear

→ need to make assumptions so that full potential becomes linear :

We assume u & v are 'small' pertubations to freestream V_∞



$$u = V_\infty + u' = V_\infty + \frac{\partial \phi'}{\partial x} \qquad \phi = V_\infty x + \phi'$$

$$v = v' = \frac{\partial \phi'}{\partial y}$$

Substituting pertubation velocity back into velocity potential equation gives "pertubation velocity potential equation":

$$\left[a^2 - \left(V_\infty + \frac{\partial \phi'}{\partial x}\right)^2\right]\frac{\partial^2 \phi'}{\partial x^2} + \left[a^2 - \left(\frac{\partial \phi'}{\partial y}\right)^2\right]\frac{\partial^2 \phi'}{\partial y^2} - 2\left(V_\infty + \frac{\partial \phi'}{\partial x}\right)\left(\frac{\partial \phi'}{\partial y}\right)\frac{\partial^2 \phi'}{\partial x \partial y} = 0$$

Ignoring products of derivatives*, we can arrive back at Laplace equation and scaling factor ahead of x 2nd derivative.

* means we are ignoring effects of non-linear features like shocks

only works below critical mach number ($\sim 0.5 - 0.6$)

Linearised velocity potential equation:

$$\boxed{\boxed{(1 - M_\infty^2)}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \approx 0}$$

we want to eliminate this term to get pure Laplace

→ Need to linearise pressure coefficient:

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2}$$

writing in velocity terms (from Bernoulli):

$$C_p = -\frac{2u'}{V_\infty} + \frac{u'^2 + v'^2}{V_\infty^2} + \text{H.O.T.}$$

$$C_p = 1 - \frac{V^2}{V_\infty^2} = 1 - \frac{(v'^2 + (V_\infty + u')^2)}{V_\infty^2} \approx -2\frac{u'}{V_\infty}$$

$$C_p \approx -\frac{2u'}{V_\infty}$$

pressure coefficient linked only to pertubation velocity in x-direction.

Subsonic Prandtl - Glauert Correction:

→ need to change scaling of plane so that constant factor dissapears:

→ this scaling transforms a compressible problem to an equivalent incompressible flow solution.

Transformed plane has scaled coordinates :

$$x_{ic} = Cx \qquad y_{ic} = Dy$$

incompressible

we also scale pertubation potential so we can map incompressible and compressible flows on the same geometry :

$$\bar{\phi}(x_{ic}, y_{ic}) = \beta \phi'(x,y)$$

$\beta = \sqrt{1-M_\infty^2}$ (Glauert factor)

→ There are two possibilities to scale x & y to transform to Laplace :

$$C = 1, \quad D = \beta \qquad \text{or} \qquad C = \frac{1}{\beta}, \quad D = 1$$

We now have Laplace equation in new coordinate system :

$$\frac{\partial^2 \bar{\phi}}{\partial x_{ic}^2} + \frac{\partial^2 \bar{\phi}}{\partial y_{ic}^2} = 0$$

the pressure coefficient becomes :

$$C_p \approx \frac{1}{\beta}\left(-\frac{2\bar{u}}{V_\infty}\right)$$

which is the incompressible pressure coefficient ÷ by β

$$\boxed{C_p \approx \frac{\bar{C}_{p,0}}{\beta}}$$

we can also find C_l & C_m as $\boxed{C_l = \frac{C_{l_0}}{\beta}}$ & $\boxed{C_m = \frac{C_{m_0}}{\beta}}$

- Predicted drag = 0 (D'Alembert)
- Aerodynamic centre (at 0.25c) unaffected

Supersonic Linearised Ackret Theory :

$$\frac{dp}{d\theta} = \frac{\gamma M^2}{\sqrt{M^2-1}} p$$

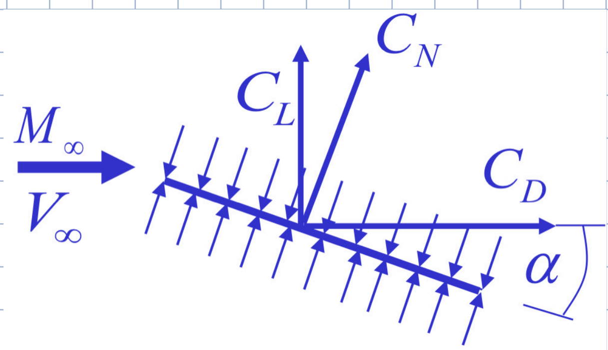
assuming $p = p_\infty$, we can integrate wrt θ

$$p - p_\infty = \frac{\gamma M^2 \theta}{\sqrt{M^2-1}} p_\infty \longrightarrow \frac{p}{p_\infty} - 1 = \frac{\gamma M^2 \theta}{\sqrt{M^2-1}}$$

$$\rightarrow \quad \boxed{C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}} \qquad \theta \text{ in radians}$$

- this relates the pressure coefficient to the angle θ the flow is turned by.
- θ +ve if normal vector outwards from surface points back along freestream and -ve if points forwards along freestream.
- this is an approximation & is quite accurate for small angles

Applying this to a flat plate 2D aerofoil at aoa α :



$$C_{p_{lower}} = \frac{2\alpha}{\sqrt{M_\infty^2 - 1}}$$

$$C_{p_{upper}} = \frac{-2\alpha}{\sqrt{M_\infty^2 - 1}}$$

$$C_N = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

$$C_l = C_N \cos\alpha \approx \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

$$C_d = C_N \sin\alpha \approx C_N \alpha \approx \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}}$$

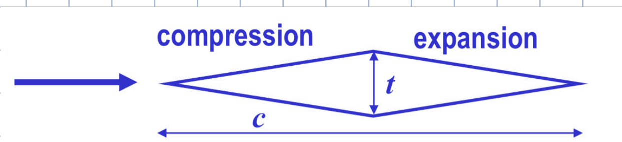
for supersonic drag $D \neq 0$ ("wave drag")

$$\boxed{C_d = \frac{\sqrt{M_\infty^2 - 1}}{4} C_l^2}$$

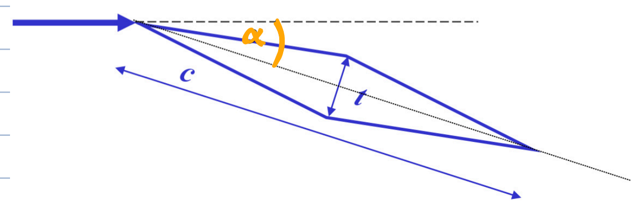
- results give a constant pressure distribution
  ↳ centre of pressure at 0.5c

For a thin 'double wedge' high speed section, linearised Ackret theory gives :

$$\boxed{C_d = 4 \frac{(t/c)^2}{\sqrt{M_\infty^2 - 1}}}$$



$$\boxed{C_d \quad 4\frac{\alpha^2 + (t/c)^2}{\sqrt{M_\infty^2 - 1}}}$$



* additional drag component due to angle of attack ($\therefore$ increased lift)

Ackeret is simpler than shock expansion theory and is useful for curved surfaces.

Wave Drag:

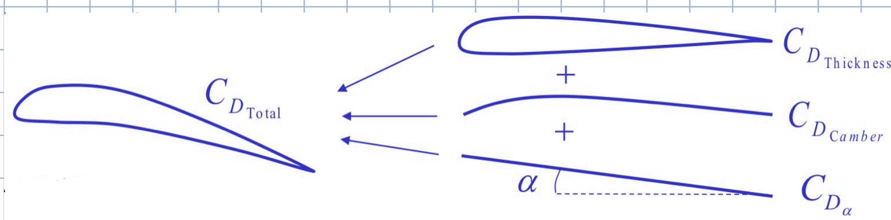
Energy is lost to wave system shed by aerofoil

- occurs even in isentropic flow

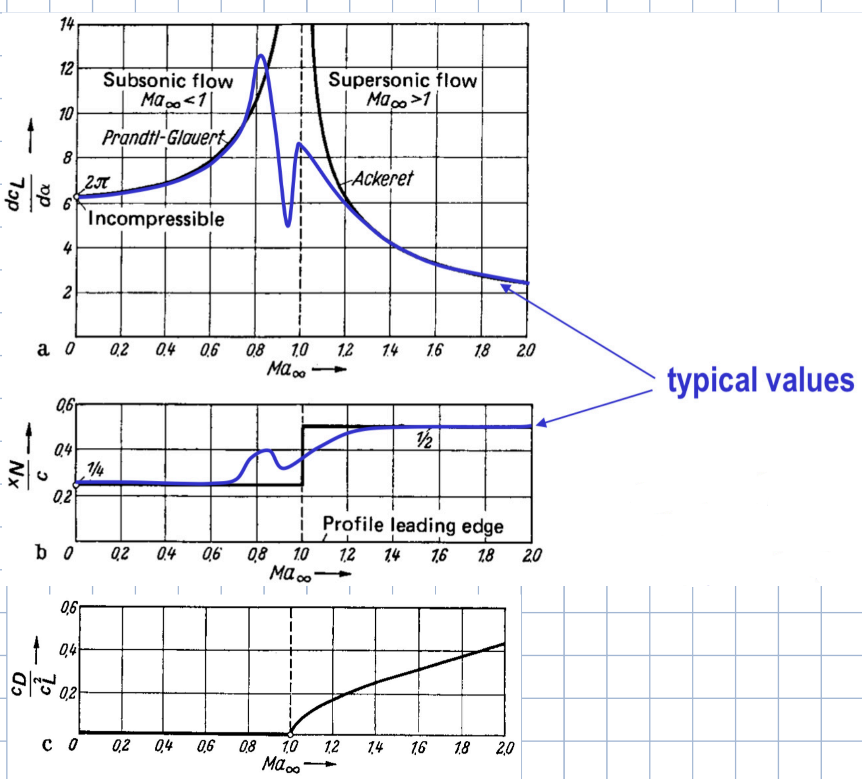
Wave drag has 3 components:

- Lift
- Thickness
- Camber

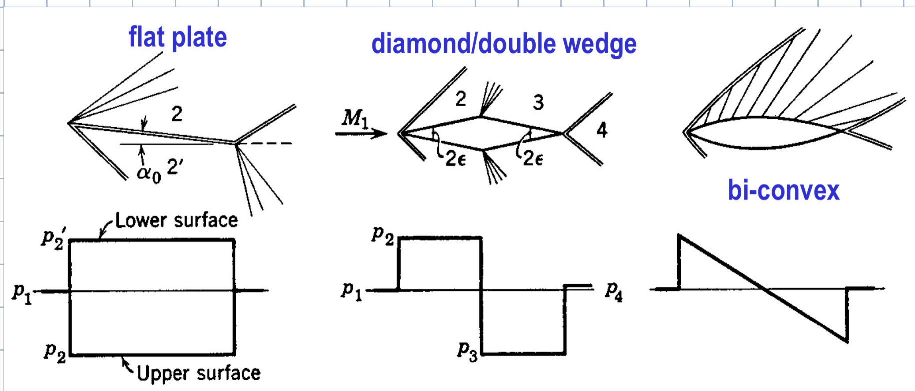
} present at zero-lift conditions



Linearised 2D Aerofoil Characteristics:



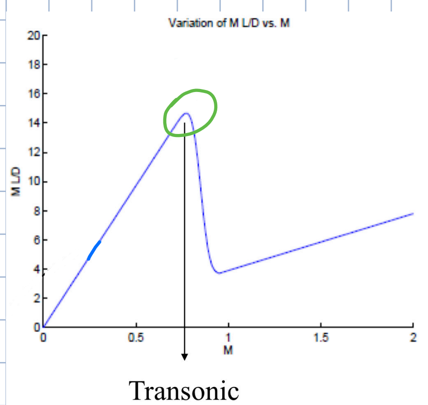
'Real' Supersonic Aerofoil Flows :

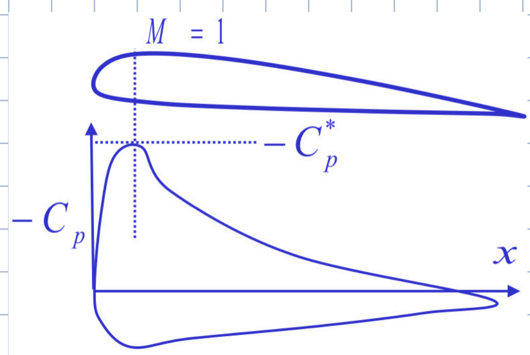


Why Fly Transonic ?

- The complex effects that arise at transonic speeds is worth the compromise to maximise the Breguet range.

→ The range is essentially proportional to $M\frac{L}{D}$ ⟶ for 'low' speed, $\frac{L}{D}$ ~ fixed





- Transonic behaviour begins when sonic flow appears on aerofoil surface.
  ↳ identification of sonic point is important

→ at this point, $M_\infty = M_{crit}$, $C_{p_{min}} = C_p{}^*$ ⟶ critical pressure coefficient

$$C_p = \left(\frac{\frac{p}{p_\infty} - 1}{\frac{1}{2}\gamma M_\infty^2} \right) \qquad \frac{p}{p_\infty} = \frac{p}{p_{0\infty}} \frac{p_{0\infty}}{p_\infty} {}^*$$

* along an isentropic streamline, p_0 is constant

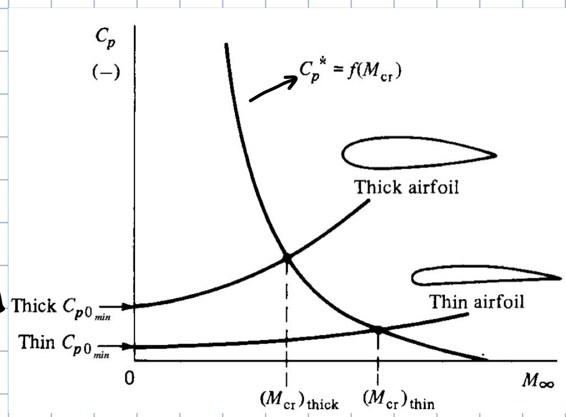
setting $M=1$ for p term

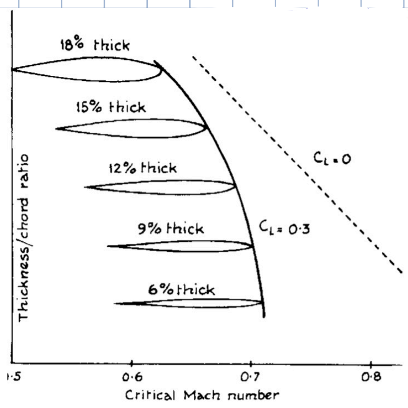
$$\frac{p}{p_\infty} = \frac{\left(1 + \frac{\gamma-1}{2} M_\infty^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(1 + \frac{\gamma-1}{2}\right)^{\frac{\gamma}{\gamma-1}}}$$

$$\therefore \quad C_p{}^* = \frac{2}{\gamma M_\infty^2} \left\{ \left(\frac{1 + 0.5(\gamma-1) M_\infty^2}{1 + 0.5(\gamma-1)} \right)^{\frac{\gamma}{\gamma-1}} - 1 \right\} = f(M_\infty)$$

- Highest local velocities correspond to minimum C_p (suction peak)
  - → depends on α
  - → generally scales directly with t/c
  - → does not necessarily occur at maximum thickness

- We approximate C_p in terms of incompressible pressure distribution C_{p_0} scaled with M_∞ by Prandtl - Glauert correction.





thicker airfoil has more -ve C_p

thinner airfoil accelerates air less $\therefore$ M_{crit_∞} is higher

Calculating C_p^*

- Calculate incompressible C_p^*

$$C_p^* = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_{0\infty}}\frac{p_{0\infty}}{p_\infty} - 1\right) \quad \therefore \quad \boxed{C_p^* = \frac{2}{\gamma M_\infty^2}\left(\frac{1}{1.895}\frac{p_{0\infty}}{p_\infty} - 1\right)} = C_{p_{min}}$$

look up this ratio in tables according to freestream mach

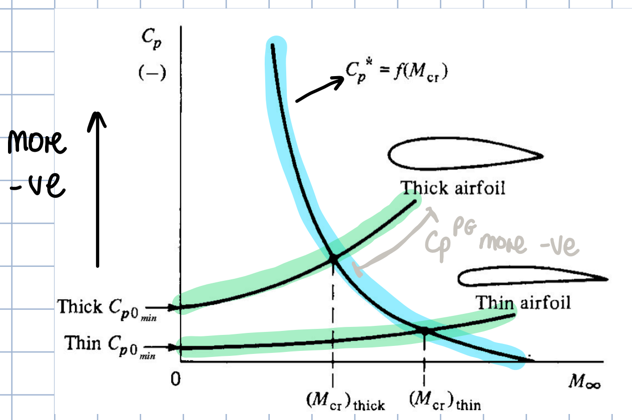
$\frac{p}{p_{0\infty}}$ is fixed sonic flow total pressure ratio for $M = 1$

$$\frac{p}{p_{0\infty}} = \frac{1}{1.895}$$

- Correcting for compressible with Prandtl - Glauert

$$\boxed{C_p^{PG} = \frac{C_{p_{min}}}{\sqrt{1-M_\infty^2}}} \longrightarrow \boxed{C_p^{comp} = \frac{C_p^{incomp}}{\sqrt{1-M_\infty^2}}}$$

- To find M_{crit}, use bisection method as shown in slides



We are trying to converge upon answer by iterating upon lower & upper limits.

→ Estimate upper and lower M values
→ Take average M value
→ calculate C_p^* and C_p^{PG} using M_{avg}

→ e.g if C_p^{PG} is more -ve than C_p^*, you know you're on the right of the intersection

↳ ∴ M value too high

↳ set M_{avg} to M_{upper} and leave M_{lower}

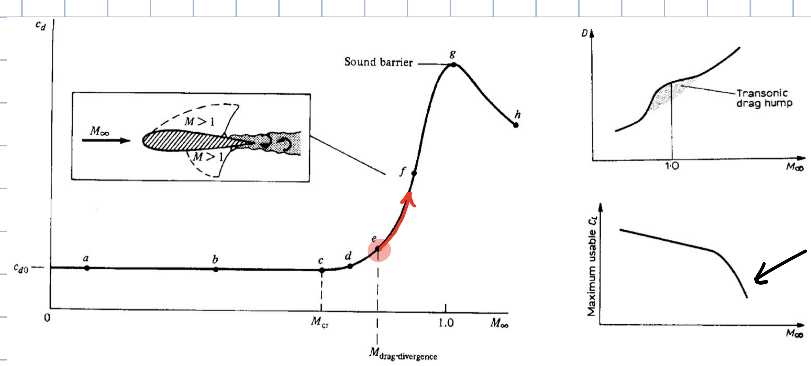
→ iterate until convergence

Are Results Accurate ?

- No 3D effects → 3D C_p less -ve ∴ ↑ M_{crit}
- Depends on incidence, as α ↑, M_{crit} ↓

Above M_{cr} :

- Drag divergence : rapid increase in drag just above M_{cr}
- Shock stall : M_{cr} decreases with α, above M_{cr} shocks form and seperation induced at shock attatchment point

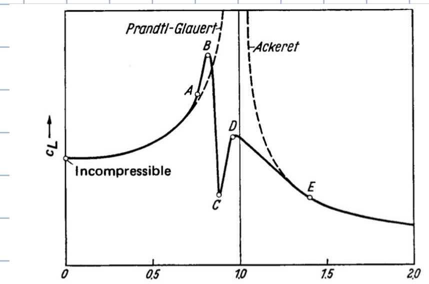


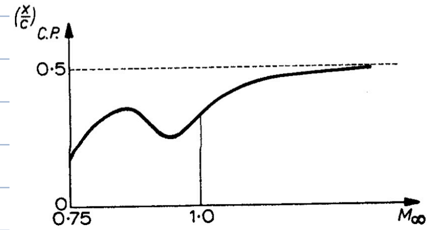
due to shocks etc. makes manoeuvres at ↑ M_∞ difficult

- Aerodynamic centre moves aft : increased stability but potentially strong nose down pitching moment

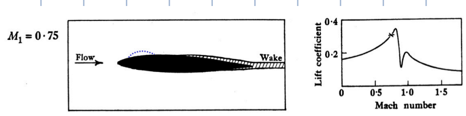
Stability determined by difference between CoM & AC

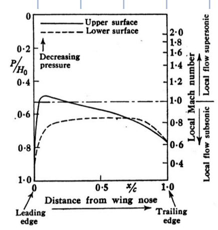
↳ CoM changed by fuel pumping





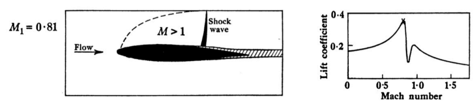
Examples of Aerofoils at Different M_∞ :

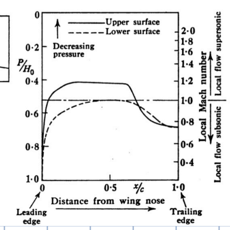




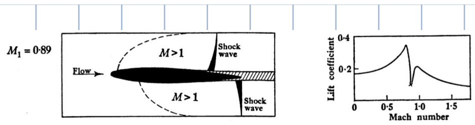
$M > M_{CR}$ →small region of supersonic flow.
Flow is still isentropic but starts to diverge from P-G solutions

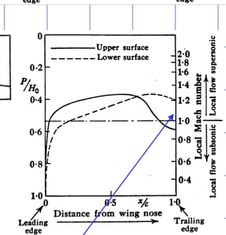
Transonic





i) supersonic region terminates with a normal shock.
ii) Aero centre moves aft
iii) Drag rise due to losses in P_o through shock



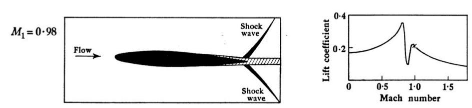


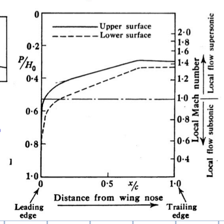
i) Supersonic flow on lower surface. Extent of supersonic region increases rapidly (flat surface).
ii) Drag continues to rise.
iii) Large lift loss
iv) Aero centre moves forward

Pressure adjusts in the wake

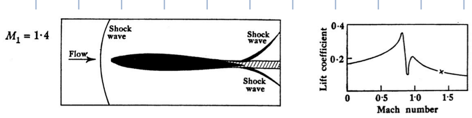
Transonic

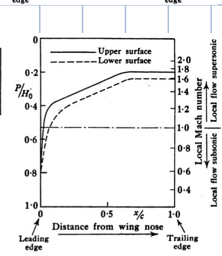
increasing M_∞





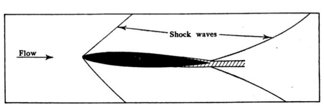
i) Shocks reach trailing edge (terminating shock system).
ii) Aero centre moves aft again
iii) Drag continues to rise.
iii) lift recovers

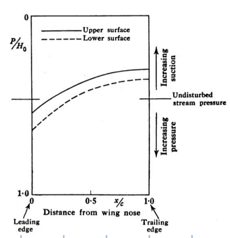




i) Bow shock forms
ii) possible terminating shocks at trailing edges
iii) Aero centre fixed at approx 0.5C

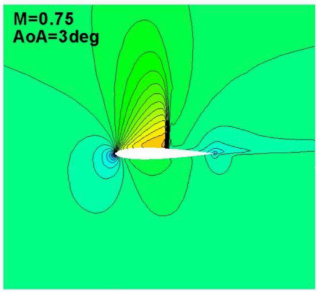
Supersonic

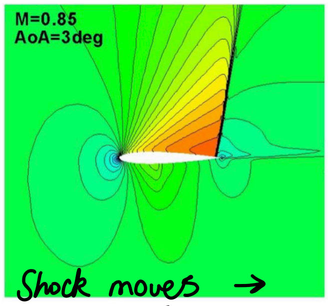




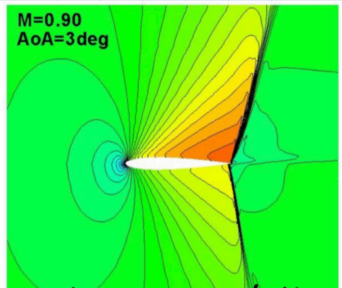
Bow shock reattaches as oblique shocks -- Mach number depends on leading edge shape.

Evolution of Shocks at $\uparrow M_\infty$:

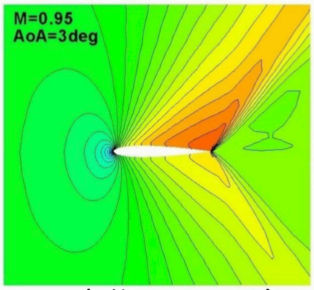




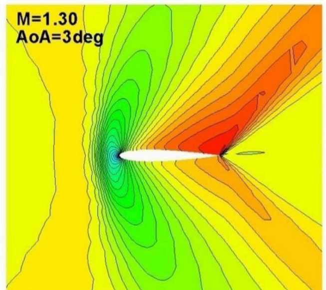
Shock moves $\rightarrow$ to TE



Shock forms on bottom



Shocks tilt over at $\uparrow M_\infty$



Bow Shock forms

Bow shock tightens itself around LE as $M_\infty \uparrow$